A field theory approach to cosmological density perturbations†

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Abstract

Adiabatic perturbations propagate in the expanding universe like scalar massless fields in some effective Robertson-Walker space-time.

1 Introduction

Sound propagation in rotationless fluid can be identified with the propagation of scalar field in some pseudo-Riemannian space. The geometry of this space is called the acoustic geometry. In the particular case of steady flows one obtains Unruh metric, which mimics Schwarzschild space-time [1]. Potentially, some basic properties of black holes can be examined in the laboratory by investigating their hydrodynamic analogues. On the other hand, several theoretical problems with moving fluids (for instance the acoustic energy problem in the expanding medium) can find their natural description in geometrical language [2].

Similar geometrization can be performed for the density perturbations in the expanding universe. This fact was noted for the radiation-dominated and spatially flat Friedman universe by Sachs, Wolfe [3], Field and Shepley [4]. The purpose of this paper is to point out that the theorem proved by Sachs-Wolfe ([3], p. 76) can be extended to all adiabatic perturbations, and to the universe of arbitrary space curvature. We demonstrate this in synchronous Lifshitz formalism. We show (in the Appendix) that all major gauge-invariant formalisms admit variables similar to the Field-Shepley H-variable, or the Sachs-Wolfe E-variable, i.e. the quantities, which propagate like the scalar field or gravitational waves on some effective Robertson-Walker background.

Throughout this paper Greek indices run from 0 to 3, while the convection $c = 1$ and $8\pi G = 1$ is used.

2 Synchronous system of reference

Consider Friedman-Robertson-Walker universe with the metric form

$$g_{\mu\nu} = a^2(\eta)\ \mathrm{diag}\left[-1,\ 1,\ \ \frac{\sin^2(\sqrt{K}\chi)}{K},\ \ \frac{\sin^2(\sqrt{K}\chi)}{K}\sin^2\vartheta\right], \tag{1}$$

and the perfect-fluid energy-momentum tensor

$$T^{\mu\nu} = [\epsilon + p(\epsilon)]\, u^\mu u^\nu + p(\epsilon)\, g^{\mu\nu}. \tag{2}$$

The equation of state $p = p(\epsilon)$, assumed here, limits investigations to the adiabatic density perturbations (see [5] section V.A). In particular dissipative processes are excluded. When the synchronous reference

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system is used the metric corrections caused by the scalar perturbations are determined by two scalar functions $\lambda(\eta)$ and $\mu(\eta)$. Both $\lambda(\eta)$ and $\mu(\eta)$ solve the system of two second order differential equations [6]

$$\lambda''(\eta) + 2\frac{a'(\eta)}{a(\eta)}\lambda'(\eta) - \frac{k^2 - K}{3}\left[\lambda(\eta) + \mu(\eta)\right] \quad = \quad 0, \tag{3}$$

$$\mu''(\eta) + \left[2 + 3\frac{p'(\eta)}{\epsilon'(\eta)}\right]\frac{a'(\eta)}{a(\eta)}\mu'(\eta) + \frac{k^2 - 4K}{3}\left[1 + 3\frac{p'(\eta)}{\epsilon'(\eta)}\right]\left[\lambda(\eta) + \mu(\eta)\right] \quad = \quad 0. \tag{4}$$

The density contrast $\delta(\eta, \mathbf{x})$ can be found in the form of Fourier integral

$$\delta(\eta, \mathbf{x}) = \int \delta_k \, Q(\mathbf{k}\cdot\mathbf{x}) \mathrm{d}^3 k + c.c. \tag{5}$$

with the Fourier coefficients [6]

$$\delta_k = \frac{1}{3\epsilon(\eta)a^2(\eta)}\left[(k^2 - 4K)\left[\lambda(\eta) + \mu(\eta)\right] + 3\frac{a'(\eta)}{a(\eta)}\mu'(\eta)\right]. \tag{6}$$

k stands for the wave number $k \equiv |\mathbf{k}|$, while $Q(\mathbf{k}\cdot\mathbf{x})$ are scalar harmonics.

We introduce a new perturbation variable $\psi(\eta, \mathbf{x})$ by employing the Darboux transformation of the density contrast

$$\psi(\eta, \mathbf{x}) = a(\eta)H^2(\eta)\frac{\partial}{\partial\eta}\frac{\delta(\eta, \mathbf{x})}{H(\eta)\left[1 + \frac{p(\eta)}{\epsilon(\eta)}\right]}. \tag{7}$$

With system (3–4) satisfied, the propagation equation for the variable $\psi(\eta, \mathbf{x})$ takes the form

$$\frac{\partial^2}{\partial\eta^2}\psi(\eta, \mathbf{x}) + \left(2\frac{\mathsf{a}'(\eta)}{\mathsf{a}(\eta)} - \frac{c_{\mathrm{s}}{}'(\eta)}{c_{\mathrm{s}}(\eta)}\right)\frac{\partial}{\partial\eta}\psi(\eta, \mathbf{x}) - c_{\mathrm{s}}{}^2(\eta)\,\triangle\psi(\eta, \mathbf{x}) = 0, \tag{8}$$

where $\triangle$ is Laplacian operating in the constant time hypersurface,

$$c_{\mathrm{s}}(\eta) = \sqrt{\frac{p'(\eta)}{\epsilon'(\eta)}} \quad \text{and} \quad \mathsf{a}(\eta) = a(\eta)\sqrt{\frac{1}{c_{\mathrm{s}}(\eta)}\frac{\epsilon(\eta) + p(\eta)}{3H^2(\eta)}}. \tag{9}$$

$c_{\mathrm{s}}(\eta)$ stands for the sound velocity, which is assumed to be strongly positive. $H(\eta) = a'(\eta)/a^2(\eta)$ stands for the Hubble parameter. Although the gauge modes may contribute to the density contrast $\delta(\eta, \mathbf{x})$, the new variable ψ is gauge invariant[1]. The equation (8) has the same structure as the Lukash [9] equation (1.10) and can be independently derived on the ground of the Lagrangian formalism.

Let us introduce a new time variable ζ defined by the integral

$$\zeta = \int c_{\mathrm{s}}(\eta)\,\mathrm{d}\eta. \tag{10}$$

The change of the time variable allows one to reduce the propagation equation for ψ to

$$\frac{\partial^2}{\partial\zeta^2}\psi(\zeta, \mathbf{x}) + 2\frac{\mathsf{a}'(\zeta)}{\mathsf{a}(\zeta)}\frac{\partial}{\partial\zeta}\psi(\zeta, \mathbf{x}) - \triangle\psi(\zeta, \mathbf{x}) = 0, \tag{11}$$

[1] The construction of the gauge invariant variables by means of Darboux transformations in a radiation dominated universe has been discussed in detail in [7], and independently employed in [8].

which is an explicit form of the d'Alambert equation

$$\Box\,\psi(\zeta,\mathbf{x}) \equiv \nabla^\mu\nabla_\mu\psi(\zeta,\mathbf{x}) = 0 \tag{12}$$

for scalar field ψ propagating in Robertson-Walker space-time

$$\mathsf{g}_{\mu\nu} = \mathsf{a}^2(\zeta)\ \mathrm{diag}\left[-1,\ 1,\ \ \frac{\sin^2(\sqrt{K}\chi)}{K},\ \ \frac{\sin^2(\sqrt{K}\chi)}{K}\sin^2\vartheta\right] \tag{13}$$

with a scale factor $\mathsf{a}(\zeta)$

$$\mathsf{a}(\zeta) = a(\zeta)\sqrt{\frac{1}{c_{\mathrm{s}}(\zeta)}\frac{\epsilon(\zeta)+p(\zeta)}{3H^2(\zeta)}}. \tag{14}$$

The equation (12) with the metric tensor (13) build the acoustic geometry, we looked for. In consequence, the adiabatic perturbations propagate as massless scalar field in the expanding universe. The new time variable ζ plays a similar role in the acoustic geometry as does the conformal time η in the original universe. This particularly concerns the shape of null cones and the sound horizons, therefore, ζ may be considered as the *acoustic conformal time*. Although the d'Alambert operator is Lorentz-invariant the propagation equation (12) is not, because of non-invariance of the field variable ψ. This limitation is characteristic of quasi-particles (phonons — in cosmological context see [10]) and well-known in the acoustic geometrical descriptions [1].

The reduction of the density perturbation equation to the d'Alambert equation, presented above, may also be understood as a generalized Sachs-Wolfe theorem ([3], p 76). Equation (12) (and in consequence (11)) is identical (up to some constant tensor factor) with the equation for gravitational wave propagating in the pressureless environment [8]. Similar equations for the density perturbations can be obtained in all major gauge invariant formalisms. Appropriate Darboux transformations for gauge-invariant equations are given in Appendix.

3 Summary

For adiabatic perturbations we constructed the perturbation variables (ψ in the synchronous system, and its analogues in gauge-invariant formalisms), which propagate like massless scalar fields (12) in some Robertson-Walker space-time. The construction of these fields and the background geometry is unique and similar in the most frequently used gauge-invariant descriptions. Since the propagation equation has been reduced to d'Alambert equation, the field theory in the curved space-time [11] become an suitable language to describe density perturbations and to determine their visual effects on the cosmic microwave background.

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Appendix

In this section we list transformations of gauge-invariant propagation equations to the form of equation (8). We limit ourselves to the transformations of perturbation variables — the time transformation in all cases is identical and equal to (10). We hold the original authors' notation where possible, yet to reach the consistency of this paper we always denote the metric scale factor by $a(t)$, the energy density by ϵ, the sound velocity by $c_{\rm s}$, the expansion rate by Hubble $H = a'(t)/a(t) = a'(\eta)/a^2(\eta)$ or the expansion scalar $\theta = 3H$. Capital K stands for the curvature index $K = 0, -1, 1$, while small k denotes the wave number.

It should be emphasized that although the quantities $\widehat{\Delta\epsilon}$, $\widehat{\Delta}$, $\widehat{\Phi_{\rm H}}$, $\widehat{\epsilon_{\rm m}}$, $\widehat{\delta\epsilon}$, $\widehat{\phi}$, below, obey the same propagation equation, they should not be identified. Geometrical differences between them are clear from how the corresponding $\Delta\epsilon$, Δ, $\Phi_{\rm H}$, $\epsilon_{\rm m}$, $\delta\epsilon$, ϕ are defined in the original papers.

Author: Olson [12] (generalized to arbitrary K in [13])
Propagation equation ([13] system (21a–21b)):

$$\frac{\partial}{\partial\eta}\Delta\epsilon(\eta,\mathbf{x}) = a(\eta)\left[-\frac{5}{3}\theta(\eta)\Delta\epsilon(\eta,\mathbf{x}) - [\epsilon(\eta)+p(\eta)]\,\Delta\theta(\eta,\mathbf{x})\right], \tag{15}$$

$$\begin{aligned}\frac{\partial}{\partial\eta}\Delta\theta(\eta,\mathbf{x}) &= a(\eta)\left[-\frac{{c_{\rm s}}^2(\eta)}{a^2(\eta)\,[\epsilon(\eta)+p(\eta)]}\,[\triangle + 3K]\,\Delta\epsilon(\eta,\mathbf{x}) - \frac{1}{2}\Delta\epsilon(\eta,\mathbf{x})\right.\\ &\qquad\left. - \frac{4}{3}\theta(\eta)\Delta\theta(\eta,\mathbf{x})\right].\end{aligned} \tag{16}$$

Variables:
$\Delta\epsilon$ — the spatial density Laplacian,
$\Delta\theta$ — the spatial Laplacian of the expansion rate.
The wave equation $\Box\,\widehat{\Delta\epsilon}(\eta,\mathbf{x}) = 0$, given explicitly by eq. (8), (equivalently the equation (12) expressed in the *acoustic conformal time*) is satisfied by:

$$\widehat{\Delta\epsilon}(\eta,\mathbf{x}) = \frac{H^2(\eta)}{a^2(\eta)\,[\epsilon(\eta)+p(\eta)]}\,\frac{\partial}{\partial\eta}\left[\frac{a^5(\eta)}{H(\eta)}\Delta\epsilon(\eta,\mathbf{x})\right]. \tag{17}$$

Authors: Bruni, Dunsby and Ellis ([14] and references given there)
Propagation equation ([14] equation (73)):

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\Delta(t,\mathbf{x}) + (2 + 3{c_{\rm s}}^2 - 6w)H\frac{\partial}{\partial t}\Delta(t,\mathbf{x}) - \left[\left(\frac{1}{2} + 4w - \frac{3}{2}w^2 - 3{c_{\rm s}}^2\right)\kappa\,\epsilon\right.\\ \left. + 12({c_{\rm s}}^2 - w)\frac{K}{a^2}\right]\Delta(t,\mathbf{x}) - {c_{\rm s}}^2\,{}^{(3)}\nabla^2\Delta(t,\mathbf{x}) = 0.\end{aligned} \tag{18}$$

Variables:
$w = p/\epsilon$ — pressure to energy ratio,
perturbation variable capital delta: $\Delta(\eta,\mathbf{x}) = a^2(\eta)\frac{\Delta\epsilon(\eta,\mathbf{x})}{\epsilon(\eta)}$.
The wave equation $\Box\,\widehat{\Delta}(\eta,\mathbf{x}) = 0$ satisfied by:

$$\widehat{\Delta}(\eta,\mathbf{x}) = \frac{H^2(\eta)}{a^2(\eta)\,[\epsilon(\eta)+p(\eta)]}\,\frac{\partial}{\partial\eta}\left[\frac{a^3(\eta)\epsilon(\eta)}{H(\eta)}\Delta(\eta,\mathbf{x})\right]. \tag{19}$$

Author: Bardeen [5]
Propagation equation (not given explicitly in the cited paper):

$$\frac{\partial^2}{\partial\eta^2}\Phi_{\mathrm{H}}(\eta,\mathbf{x})+\left[1+{c_{\mathrm{s}}}^2(\eta)\right]a(\eta)\theta(\eta)\frac{\partial}{\partial\eta}\Phi_{\mathrm{H}}(\eta,\mathbf{x})$$
$$-\left[2K\left[1+3{c_{\mathrm{s}}}^2(\eta)\right]+a^2(\eta)\epsilon(\eta)\left[\frac{p(\eta)}{\epsilon(\eta)}-{c_{\mathrm{s}}}^2(\eta)\right]+{c_{\mathrm{s}}}^2(\eta)\triangle\right]\Phi_{\mathrm{H}}(\eta,\mathbf{x})=0. \tag{20}$$

Variables:
Φ_{H} — gauge invariant potential.
The wave equation $\Box\,\widehat{\Phi_{\mathrm{H}}}(\eta,\mathbf{x})=0$ satisfied by

$$\widehat{\Phi_{\mathrm{H}}}(\eta,\mathbf{x})=\frac{H^2(\eta)}{a^2(\eta)\left[\epsilon(\eta)+p(\eta)\right]}\frac{\partial}{\partial\eta}\left[\frac{a(\eta)}{H(\eta)}\Phi_{\mathrm{H}}(\eta,\mathbf{x})\right]. \tag{21}$$

Author: Bardeen [5]
Propagation equation ([5] equation (4.9))

$$\frac{\partial^2}{\partial\eta^2}\left[\epsilon(\eta)a^3(\eta)\epsilon_{\mathrm{m}}(\eta,\mathbf{x})\right]+\left[1+3{c_{\mathrm{s}}}^2(\eta)\right]\frac{a'(\eta)}{a(\eta)}\frac{\partial}{\partial\eta}\left[\epsilon(\eta)a^3(\eta)\epsilon_{\mathrm{m}}(\eta,\mathbf{x})\right]$$
$$+\left[-{c_{\mathrm{s}}}^2(\eta)\left[\triangle+3K\right]-\frac{1}{2}\left[\epsilon(\eta)+p(\eta)\right]a^2(\eta)\right]\left[\epsilon(\eta)a^3(\eta)\epsilon_{\mathrm{m}}(\eta,\mathbf{x})\right]=0. \tag{22}$$

Variables:
ϵ_{m} — the density contrast measured on the flow-orthogonal surfaces.
The wave equation $\Box\,\widehat{\epsilon_{\mathrm{m}}}(\eta,\mathbf{x})=0$ satisfied by:

$$\widehat{\epsilon_{\mathrm{m}}}(\eta,\mathbf{x})=\frac{H^2(\eta)}{a^2(\eta)\left[\epsilon(\eta)+p(\eta)\right]}\frac{\partial}{\partial\eta}\left[\frac{a^3(\eta)\epsilon(\eta)}{H(\eta)}\epsilon_{\mathrm{m}}(\eta,\mathbf{x})\right]. \tag{23}$$

Authors: Lyth and Stewart ([15] and references given there)
Propagation equation ([15], equations (35–36))

$$\frac{\partial}{\partial t}\delta\epsilon(t,\mathbf{x}) = -\theta(t)\delta\epsilon(t,\mathbf{x})-\left[\epsilon(t)+p(t)\right]\delta\theta(t,\mathbf{x}), \tag{24}$$
$$\frac{\partial}{\partial t}\delta\theta(t,\mathbf{x}) = -\left[\frac{1}{2}+\frac{{c_{\mathrm{s}}}^2(\eta)\left[3\epsilon(t)-\theta^2(t)\right]}{3\left[\epsilon(t)+p(t)\right]}\right]\delta\epsilon(t,\mathbf{x})-\frac{2}{3}\theta(t)\delta\theta(t,\mathbf{x})$$
$$-\frac{{c_{\mathrm{s}}}^2(\eta)}{\epsilon(t)+p(t)}\triangle\delta\epsilon(t,\mathbf{x}). \tag{25}$$

Variables:
$\delta\epsilon$, $\delta\theta$ — variations of density and the expansion rate on the flow-orthogonal hypersurfaces.
The wave equation $\Box\,\widehat{\delta\epsilon}(\eta,\mathbf{x})=0$ satisfied by:

$$\widehat{\delta\epsilon}(\eta,\mathbf{x})=\frac{H^2(\eta)}{a^2(\eta)\left[\epsilon(\eta)+p(\eta)\right]}\frac{\partial}{\partial\eta}\left[\frac{a^3(\eta)\epsilon(\eta)}{H(\eta)}\delta\epsilon(\eta,\mathbf{x})\right]. \tag{26}$$

Authors: Brandenberger, Kahn and Press [16] (generalized to arbitrary K in [17])
Propagation equation (longitudinal gauge, [17] equation (5.3) for $\delta S = 0$.)

$$\begin{aligned}\phi''(\eta,\mathbf{x}) \quad &+ \quad 3(1+{c_\mathrm{s}}^2(\eta))\mathcal{H}(\eta)\phi'(\eta,\mathbf{x}) - {c_\mathrm{s}}^2(\eta)\triangle\phi(\eta,\mathbf{x}) \\ &+ \quad \left[2\mathcal{H}'(\eta) + (1+3{c_\mathrm{s}}^2(\eta))(\mathcal{H}^2(\eta)-K)\right]\phi(\eta,\mathbf{x}) = 0. \end{aligned} \tag{27}$$

Variables:
$\mathcal{H}(\eta) = a'(\eta)/a(\eta)$,
$\phi(\eta,\mathbf{x})$ — gauge-invariant potential.
The wave equation $\Box\,\widehat{\phi}(\eta,\mathbf{x}) = 0$ satisfied by:

$$\widehat{\phi}(\eta,\mathbf{x}) = \frac{1}{\epsilon(\eta)+p(\eta)}\left[\frac{\mathcal{H}(\eta)}{a^2(\eta)}\right]^2 \frac{\partial}{\partial\eta}\left[\frac{a^2(\eta)}{\mathcal{H}(\eta)}\phi(\eta,\mathbf{x})\right]. \tag{28}$$

References

[1] W. Unruh, Phys. Rev. Lett. **46** (1981) 1351,
W. Unruh, Phys. Rev. D **51** (1995) 2827,
M. Visser, Class. Quantum Grav. **15** (1998) 1765,
N. Bilic, Class. Quantum Grav. **16** (1999) 3953.

[2] M. Stone, Phys. Rev. D **62** (2000) 1341.

[3] R. K. Sachs and A. M. Wolfe, Astrophys. J. **147** (1967) 73,

[4] G. B. Field and L. C. Shepley, Astroph. Space Sci. **1** (1968) 309.

[5] J. M. Bardeen, Phys. Rev. D **22** (1980) 1882.

[6] E. M. Lifshitz and I. M. Khalatnikov, Adv. Phys. **12** (1963) 185.

[7] Z. A. Golda and A. Woszczyna, J. Math. Phys. **42** (2001) 456.

[8] L. P. Grishchuk, Phys. Rev. D **50** (1994) 7151.

[9] V. N. Lukash, Sov. Phys. JETP **79** (1980) 1601.

[10] G. V. Chibisov, V. F. Mukhanov, Mon. Not. R. Astron. Soc. **200** (1982) 535.

[11] N. D. Birrell and P. C. W. Davies *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).

[12] D. W. Olson, Phys. Rev. D **14** (1976) 327.

[13] A. Woszczyna and A. Kulak, Class. Quantum Grav. **6** (1989) 1665.

[14] M. Bruni, P. K. S. Dunsby and G. F. R. Ellis, Astrophys. J. **395** (1992) 34.

[15] D. Lyth and E. Stewart, Astrophys. J. **361** (1990) 343.

[16] R. Brandenberger, R. Kahn, and W. Press, Phys. Rev. D **28** (1983) 1809.

[17] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rep. **215** (1992) 203.